J. Fluid Mech. (1970), vol. 41, part 4, pp. 865–872
Printed in Great Britain

On the precession of a resonant cylinder

By ROGER F. GANS
Department of Geology, University of California, Los Angeles†

(Received 23 July 1969)

A fluid contained in a rotating cylinder has an inertial mode which is excited by forced precession of the container. Wood's (1965, 1966) early work specifically excluded resonance phenomena. Recently McEwan (1970) has discussed resonance phenomena for strong amplitude of excitation, corresponding to rapid precession in this work.

In this paper the magnitude of the resonant response for small precession rate is precisely calculated by matching the Ekman layer suction to the precessional forces. The procedure is to find the resonant mode $\mathbf{v}_1$, compute its boundary layers, $\tilde{\mathbf{v}}_1$, and the associated Ekman layer suction. The second-order problem has a solvability condition which is satisfied by matching the Ekman layer suction to the precession.

1. Introduction

The condition that a rotating cylinder filled with an inviscid fluid be resonant under a precessional motion, that is, that there exists an inertial mode with the frequency of rotation, is such that any cylinder is arbitrarily close to resonance. Wood's (1965, 1966) original work left open the question of resonance. Recently McEwan (1969) has performed a series of experiments confirming the existence of resonance. He has estimated the magnitude of the response for large excitation amplitude assuming a non-linear limiting mechanism. In this paper a precise calculation of the response amplitudes, valid for small excitation amplitude, is made, and the results compared with experiment.

The plan of this paper is to seek solutions of the Navier–Stokes equations inside a cylinder of length L and diameter D rotating about its symmetry axis with angular velocity ω and precessing about some other axis with angular velocity Ω. The direction of the precession axis is arbitrary. It will be taken to be perpendicular to the rotation axis, and it will be supposed that the viscosity is small enough to justify a boundary-layer approach. First, the resonance being considered will be shown to be a necessary consequence of seeking a response of the size $\Omega/\omega \ll 1$. Then a set of inertial modes will be found, and matched to the container boundary by a boundary layer. This layer will be seen to be an Ekman layer with an associated Ekman suction of size $\epsilon E^{\frac{1}{2}}$. Matching the suction corresponding to the resonant mode to the precessional forces in an integral sense gives $\epsilon \sim E^{-\frac{1}{2}}(\Omega/\omega)$. The non-resonant modes decay and are unimportant for the

† Present address: Division of Geological Sciences, California Institute of Technology.

steady-state response being considered. The matching condition can be shown to be correct for $\Omega/\omega < E^{\frac{3}{4}}$.

Experiments were performed for precession rates spanning the range of validity of the theory. They exhibit agreement with theory in the range where agreement is expected. Beyond this range an interesting non-stationary behaviour is imposed on a flow which is qualitatively like that given by theory.

2. Formulation

Consider a cylinder of length L and diameter D rotating about its symmetry axis with angular velocity $\boldsymbol{\omega}$, and precessing about an axis at right angles to $\boldsymbol{\omega}$ with angular velocity $\boldsymbol{\Omega}$. The container is filled with an incompressible fluid with kinematic viscosity ν. The reference co-ordinate system is assumed to rotate with $\boldsymbol{\Omega}$, and is defined to be such that

$$\boldsymbol{\omega} = \omega\mathbf{k}; \quad \boldsymbol{\Omega} = \Omega\mathbf{i}.$$

Using $\mathbf{v}$ for velocity, P for pressure and ρ for density, the appropriate equations are

$$\left.\begin{aligned} \mathbf{v}.\nabla\mathbf{v} + 2\Omega\mathbf{i}\times\mathbf{v} - \nu\nabla^2\mathbf{v} + \nabla\{P/\rho + \tfrac{1}{2}\Omega^2(x^2+y^2)\} &= 0,\\ \nabla.\mathbf{v} &= 0. \end{aligned}\right\} \tag{2.1}$$

In this reference frame steady-state solutions are possible and the time derivative of $\mathbf{v}$ has been dropped. The appropriate boundary condition is $\mathbf{v} = \boldsymbol{\omega}\times\mathbf{r}$. Non-dimensionalizing according to the scheme

$$\mathbf{r} = \tfrac{1}{2}D\mathbf{r}', \quad \mathbf{v} = \tfrac{1}{2}\omega D\mathbf{v}', \quad (P/\rho) + \tfrac{1}{2}\Omega^2(x^2+y^2) = \tfrac{1}{4}\omega^2D^2P'$$

produces the equations

$$\left.\begin{aligned} \mathbf{v}'.\nabla\mathbf{v}' + 2R_p\mathbf{i}\times\mathbf{v}' - E\nabla^2\mathbf{v}' + \nabla P' &= 0,\\ \nabla.\mathbf{v}' &= 0, \end{aligned}\right\} \tag{2.2}$$

where $R_p = \Omega/\omega$ and $E = 4\nu/\omega D^2$. The primes serve no further use and can be dropped. The boundary condition becomes $\mathbf{v} = \mathbf{k}\times\mathbf{r}$.

These equations will be solved under the assumption that R_p and E are both small, using the boundary-layer expansion

$$\left.\begin{aligned} \mathbf{v} &= \mathbf{k}\times\mathbf{r} + \epsilon_1\mathbf{v}_1 + \epsilon_2\mathbf{v}_2 + \ldots,\\ &\qquad + \epsilon_1\tilde{\mathbf{v}}_1 + \epsilon_2\tilde{\mathbf{v}}_2 + \ldots,\\ P &= \tfrac{1}{2}(x^2+y^2) + \epsilon_1P_1 + \epsilon_2P_2 + \ldots,\\ &\qquad + \epsilon_1\tilde{P}_1 + \epsilon_2\tilde{P}_2 + \ldots, \end{aligned}\right\} \tag{2.3}$$

where the boundary-layer functions, exponentially small away from the boundary, are denoted by a tilde. The ϵ's are $\epsilon_2 \ll \epsilon_1 \ll 1$. It will be shown that a consistent ordering is $\epsilon_1 \sim R_pE^{-\frac{1}{2}}$ and $\epsilon_2 \sim R_p$.

The resonance problem can be demonstrated by trying to put $\epsilon_1 \sim R_p$. The interior first-order equations become

$$\left.\begin{aligned} [\Delta - 4(\partial^2/\partial z^2)]\,P_1 &= 0,\\ [(2/\varpi) + (\partial/\partial\varpi)]\,P_1 &= 0, \quad \text{on} \quad \varpi(=\sqrt{(x^2+y^2)}) = 1\\ \partial P_1/\partial z &= -2\varpi\,e^{i\phi}, \quad \text{on} \quad z = \pm L/D. \end{aligned}\right\} \tag{2.4}$$

P_1 and $\mathbf{v}_1$ have been taken proportional to $e^{i\phi}$ and the equations written in a cylindrical co-ordinate system ϖ, ϕ, z.

The solution for P_1 is

$$P_1 = \sum_l A_l J_1(k_l \varpi) \sin(k_l/\surd 3)\, z\, e^{i\phi}.$$

The first boundary condition requires that $\{k_l\}$ satisfy

$$2J_1(k_l) + k_l J_1'(k_l) = 0. \tag{2.5}$$

Using this relation as the basis for a Dini series (Erdelyi *et al.* 1953, pp. 71–2) one can expand ϖ and find the condition for the $\{A_l\}$, viz.

$$\frac{k_l}{\surd 3} A_l \cos \frac{k_l}{\surd 3} \frac{L}{D} = -\frac{4k_l J_2(k_l)}{(3+k_l^2)\,[J_1(k_l)]^2}, \tag{2.6}$$

and this is adequate so long as $\cos[(k_l/\surd 3)(L/D)]$ does not vanish. The condition that it vanish is that

$$\frac{L}{D} - \frac{(2n+1)\pi\surd 3}{2k_l} = 0, \tag{2.7}$$

and for any L/D one can choose n and l so that the left-hand side of (2.7) is arbitrarily close to zero.

It will be supposed that $\mathbf{v}_1$ represents the resonant response and $\mathbf{v}_2$ the non-resonant. The ϵ's will be chosen as $\epsilon_1 = R_p E^{-\frac{1}{2}}$ and $\epsilon_2 = R_p$. This choice will be justified below. The procedure is as follows:

(1) find the resonant mode;

(2) find the boundary layers required;

(3) compute the Ekman layer suction; and

(4) compute the amplitude of $\mathbf{v}_1$ by matching the Ekman layer suction with the $2R_p \mathbf{i} \times \mathbf{v}_0$ term.

This procedure is analogous to that used by Busse (1968) in his investigation of the precessing spheroid, which is resonant for zero ellipticity.

3. The first-order solutions

The interior solutions are governed by the homogeneous equivalent of (2.4). Only the resonant term will be driven and

$$P_1 = A_l J_1(k_l \varpi) \sin(k_l/\surd 3)\, z\, e^{i\phi}, \tag{3.1}$$

where l is the index identifying the particular resonance. If one writes $\mathbf{v}_1$ in cylindrical co-ordinates as (u_1, v_1, w_1), these components in terms of P_1 are

$$\left.\begin{aligned} u_1 &= -\frac{i}{3}\left\{\frac{2}{\varpi} + \frac{\partial}{\partial \varpi}\right\} P_1, \\ v_1 &= \frac{1}{3}\left\{2\frac{\partial}{\partial \varpi} + \frac{1}{\varpi}\right\} P_1, \\ w_1 &= i\frac{\partial P_1}{\partial z}. \end{aligned}\right\} \tag{3.2}$$

Only the normal components vanish on the boundary so a boundary-layer solution $(\tilde{u}_1, \tilde{v}_1, \tilde{w}_1)$ such that

$$u_1 + \tilde{u}_1 = v_1 + \tilde{v}_1 = w_1 + \tilde{w}_1 = 0,$$

on the boundary is required. Since u_1 and w_1 both vanish on portions of the boundary, one requires a different treatment of the boundary-layer problem on the two different sorts of boundaries.

On the end-plates rescale in the z direction so that $\zeta = \pm(z \mp L/D)E^{-\frac{1}{2}}$, and put $\tilde{\mathbf{v}}_1$ proportional to $\exp[(\zeta/\delta) + i\phi]$. The divergence condition to lowest order is $\partial\tilde{w}_1/\partial\zeta = 0$. Since $\tilde{w}_1$ must vanish in the interior, $\tilde{w}_1 = 0$. The $\mathbf{k}$ component of the momentum equation to lowest order is $\partial\tilde{P}_1/\partial\zeta = 0$, and the same argument gives $\tilde{P}_1 = 0$. The remaining two equations give

$$\delta_1 = \frac{1+i}{\sqrt{2}}, \quad \delta_2 = \frac{1-i}{\sqrt{6}}.$$

If
$$\tilde{u}_1 = A_1 \exp(\zeta/\delta_1 + i\phi) + A_2 \exp(\zeta/\delta_2 + i\phi)$$
and
$$\tilde{v}_1 = iA_1 \exp(\zeta/\delta_1 + i\phi) - iA_2 \exp(\zeta/\delta_2 + i\phi)$$
are substituted into the boundary condition, the result, after some algebraic manipulation, and simplification of Bessel function expressions, is

$$\left.\begin{aligned} \tilde{u}_1 &= \pm[\tfrac{1}{2}ik_l J_0(k_l\varpi)\,e^{\zeta/\delta_1} + \tfrac{1}{6}ik_l J_2(k_l\varpi)\,e^{\zeta/\delta_2}]\,A_l e^{i\phi}, \\ \tilde{v}_1 &= \pm[-\tfrac{1}{2}k_l J_0(k_l\varpi)\,e^{\zeta/\delta_1} + \tfrac{1}{6}k_l J_2(k_l\varpi)\,e^{\zeta/\delta_2}]\,A_l e^{i\phi}. \end{aligned}\right\} \tag{3.3}$$

Because of the non-axisymmetric nature of the problem it is necessary to scale the radial co-ordinate by $E^{\frac{1}{2}}$, rather than $E^{\frac{1}{3}}$ or $E^{\frac{1}{4}}$. Thus one puts $\eta = (1-\varpi)E^{-\frac{1}{2}}$ and puts $\tilde{\mathbf{v}}_1$ proportional to $\exp(\eta/\delta + i\phi)$. The normal velocity and first-order pressure are eliminated as above, and the boundary-layer equations are

$$\{i - (1/\delta^2)\}(\tilde{v}_1, \tilde{w}_1) = 0, \tag{3.4}$$

and $\delta^2 = -i$. This single δ is

$$\delta = -(1-i)/\sqrt{2} = -\sqrt{3}\,\delta_2. \tag{3.5}$$

The solution is then written in the form

$$\tilde{v}_1 = A \exp(-\eta/\sqrt{3}\,\delta_2 + i\phi); \quad \tilde{w}_1 = -i(\partial A/\partial z)\exp(-\eta/\sqrt{3}\,\delta_2 + i\phi).$$

The boundary condition determines

$$A = J_1(k_l)\sin(k_l/\sqrt{3})\,zA_l,$$

and so the value of $\tilde{w}_1$, on the boundary, is

$$-i(k_l/\sqrt{3})\,J_1(k_l)\cos(k_l/\sqrt{3})\,zA_l,$$

and (3.2) shows that $w_1 + \tilde{w}_1 = 0$ on $\varpi = 1$ as required.

The Ekman layer suction associated with $\tilde{\mathbf{v}}_1$ is obtained by integrating the three divergence conditions

$$\left.\begin{aligned} \frac{\partial}{\partial\zeta}\tilde{w}_2 + \frac{1}{\varpi}\frac{\partial}{\partial\varpi}(\varpi\tilde{u}_1) + \frac{i}{\varpi}\tilde{v}_1 &= 0, \\ \frac{\partial}{\partial\eta}\tilde{u}_2 + \frac{i}{\varpi}\tilde{v}_1 + \frac{\partial}{\partial z}\tilde{w}_1 &= 0. \end{aligned}\right\} \tag{3.6}$$

This is a straightforward integration and the result is

$$\left.\begin{aligned}\tilde{w}_2 &= i\frac{k_l^2}{6}J_1(k_l\varpi)(\delta_2-3\delta_1)A_l e^{i\phi},\\ \tilde{u}_2 &= -i\sqrt{3}\,J_1(k_l)\,\delta_2\left[1+\frac{k_l^2}{3}\right]\sin\frac{k_l}{\sqrt{3}}zA_l e^{i\phi}.\end{aligned}\right\}\tag{3.7}$$

4. The second-order problem

To match the Ekman layer suction (3.7) above, a second-order interior solution is required. Neglecting terms of $O(\epsilon_1^2)$ and higher gives

$$\left.\begin{aligned}2\mathbf{k}\times\mathbf{v}_2+(\partial/\partial\phi)\,\mathbf{v}_2+\nabla P_2 &= 2\mathbf{i}\times(\mathbf{k}\times\mathbf{r}),\\ \nabla\,.\,\mathbf{v}_2 &= 0,\end{aligned}\right\}\tag{4.1}$$

as the equations of motion. Putting $\mathbf{v}_2 = (u_2, v_2, w_2)$ leads to the same representation in terms of P_2 with the addition of an inhomogeneity, e.g.

$$u_2 = -\frac{i}{3}\left\{\frac{\partial}{\partial\varpi}+\frac{2}{\varpi}\right\}P_2.\tag{4.2}$$

In terms of P_2 the boundary-value problem is

$$\left.\begin{aligned}[\Delta-4(\partial^2/\partial z^2)]\,P_2 &= 0,\\ -\frac{i}{3}\left[\frac{\partial}{\partial\varpi}+\frac{2}{\varpi}\right]P_2 &= -\tilde{u}_2 \quad\text{on}\quad \varpi = 1,\\ i\,(\partial/\partial z)\,P_2 &= -\tilde{w}_2-2i\varpi\,e^{i\phi}\quad\text{on}\quad z = \pm L/D.\end{aligned}\right\}\tag{4.3}$$

A non-trivial solution for the homogeneous equivalent of this problem exists. (It is, in fact, just of the form (3.1).) The condition that this problem have a solution is obtained by multiplying by the conjugate of the homogeneous solution and integrating over the volume. In doing this one makes use of the boundary conditions. These involve the suction terms, expressed in terms of $\mathbf{v}_1$, and hence A_l, and so the solvability condition balances the suctions against the forcing term and determines A_l. After some simplification the result, for the primary resonance $n = 0$, is

$$\begin{aligned}\int_0^1[J_1(k_l\varpi)]^2\,\varpi\,d\varpi\,[k_l^2(\delta_2-3\delta_1)]\,A_l+12\int_0^1 J_1(k_l\varpi)\,\varpi^2\,d\varpi\\ +3\sqrt{3}\,[J_1(k_l)]^2\,\delta_2(1+\tfrac{1}{4}\pi^2)\int_{-L/D}^{L/D}\sin^2\frac{k_l}{\sqrt{3}}z\,dz\,A_l = 0.\end{aligned}\tag{4.4}$$

The Bessel function integrals can be found in Erdelyi *et al.* (1953, p. 71, p. 45) and using the eigenvalue relation one obtains for A_l the expression

$$A_l = \frac{72}{\{\delta_2[3+k_l^2+6\sqrt{3}(1+\frac{1}{4}\pi^2)]-3\delta_1(3+k_l^2)\}\,k_l^2J_1(k_l)}.\tag{4.5}$$

Resonance is important for A_l large compared to $E^{\frac{1}{2}}$. For large l,

$$A_l \sim [k_lJ_1(k_l)]^{-1} \sim k_l^{-\frac{7}{2}}.$$

For resonance to be important, then

$$k_l < E^{-\frac{1}{7}}, \tag{4.6}$$

which is a severe restriction for most laboratory situations. The first few k_l are

$$k_l = 2{\cdot}7202,\ 5{\cdot}6912,\ 8{\cdot}7665,\ 11{\cdot}8752.$$

and their primary ($n = 0$) resonances are

$$(L/D)_{\text{res}} = 1,\ 0{\cdot}478,\ 0{\cdot}310,\ 0{\cdot}229.$$

Note that the lowest-order non-linear term, of order ϵ_1^2, is either axisymmetric or doubly periodic in the azimuthal co-ordinate. Thus any such terms in the integral (4.4) would vanish. In other words, the second-order non-linear terms are automatically orthogonal to the forcing function, and the results given above are valid to $O(\epsilon_1^2)$. The maximum precession rate for which the theory is valid is given by a comparison of cubic non-linearities and viscosity, namely

$$\Omega < (4\nu/\omega D^2)\,\omega.$$

5. Discussion and experiment

The flow can best be described in terms of a neutral curve—a curve on which the velocity is zero. This plays a role similar to the tipped rotation axis for a precessing spheroid. From (3.2) one can see that u_1 and w_1 are in phase and vanish together on the surface

$$\tan\phi_0 = -\frac{\operatorname{Im} A_l}{\operatorname{Re} A_l}. \tag{5.1}$$

On this plane the azimuthal velocity is

$$v_\phi = \varpi + \tfrac{1}{3}R_p E^{-\frac{1}{2}}|A_l|\,[2k_l J_1'(k_l\varpi) + (1/\varpi)\,J_1(k_l\varpi)]\sin(k_l/\surd 3)\,z + O(R_p).$$

Since $\varpi \ll 1$ on the neutral curve, one can obtain a good approximation to the curve by putting $v_\phi = 0$ and replacing the Bessel functions by the leading terms in their power series expansion. This gives

$$\varpi = R_p E^{-\frac{1}{2}}(k_l/2)\,|A_l|\sin(k_l/\surd 3)\,z + O(R_p^3 E^{-\frac{3}{2}}). \tag{5.2}$$

To test the theory one wants to maximize the response while satisfying (4.6) and (4.7). These are conflicting goals, as can be seen by noting that the response is $\sim R_p E^{-\frac{1}{2}}$ and (4.7) requires that $R_p < E^{\frac{3}{4}}$, hence the linear response must be bounded by $E^{\frac{1}{4}}$. Thus, the better the linear conditions are satisfied, the smaller the response.

An experiment was performed in an acrylic cylinder mounted horizontally and driven through a gear belt with an 1800 rev/min electric motor. The cylinder was mounted on an air-bearing table which could be precessed about a vertical axis by a continuously variable electric transmission. Precession rates from 0 to 80 rev/min were attainable, measured by timing several revolutions of the table. The relevant experimental parameters were:

$$L = D = 15\,\text{cm},$$

$$\omega = 1800\,\text{rev/min},$$

$$\nu = 400\,\text{cSt (glycerine:water} = 96{:}4).$$

These parameters give $E = 4 \times 10^{-4}$. The inequality (4.6) is near equality, but for $l = 1$ this is not important. The condition (4.7) is violated for $\Omega > 5\,\text{rev/min}$, but this does not appear to be a difficulty.

The theoretical angle ϕ_0 is 85°. This could not be measured accurately but one could estimate that the observed angle was between 70° and 90°.

The theoretical position of the neutral curve on the end-plate was

$$\varpi_{\max} = 1{\cdot}50\,\text{mm}\,(\text{rev/min})^{-1}.$$

This was checked by mounting a disk with concentric circles 2 mm apart on the inside of the container. Figure 1 shows the theoretical curve, data points, and a least-squares fit to the data. The slope of the least-squares fit is $1{\cdot}47$ mm $(\text{rev/min})^{-1}$, agreeing with the theoretical estimate, but the response at zero rev/min is 0·7 mm according to the least-squares fit. This zero crossing error is probably a reflexion of the difficulty of projecting the location of the neutral curve across the boundary layer of $O(5\,\text{mm})$, inaccuracies in the circles, an upward displacement of the bubbles caused by gravity, and general optical distortion.

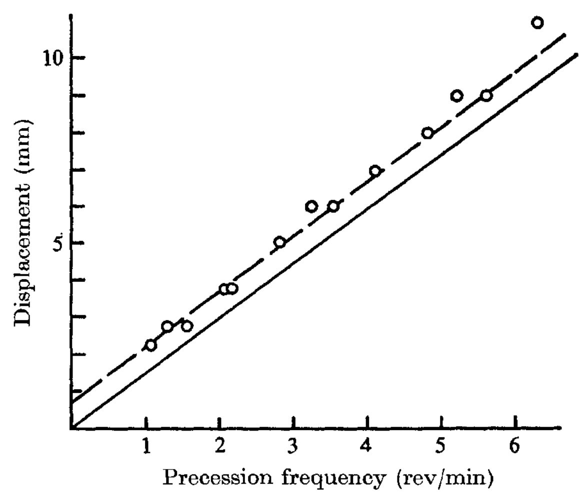

FIGURE 1. Displacement of the neutral curve *vs.* precession frequency: $\omega = 1800$ rev/min, $R = 75$ mm, $E = 4 \times 10^{-4}$. Circles are data points, solid line is theory and dashed line is least-squares fit to data.

Thus the resonance problem is resolved and the amplitude is seen to be limited by suction from non-axisymmetric Ekman layers. It has been further shown that resonance is only important for the first few L/D for Ekman numbers attainable in the laboratory.

The behaviours of these flows beyond the range of the linear theory will be discussed in detail at another time. Some runs were made with water in place of the glycerine–water mixture. For this case $E = 10^{-6}$ and the linear limit is $\Omega \sim 0{\cdot}1\,\text{rev/min}$. For Ω large enough so that significant displacements of the neutral curve took place, the flow was not sufficiently stable to make measurements. The average position of the neutral curve seemed to be about half that which would be predicted by the linear theory. An average ϕ would be less than 45°. The axis was observed to pitch about, and to shed vortices which stabilized at about two-thirds of the way to the side wall, were stable there for several seconds and then collapsed toward the centre, to be replaced by newly generated vortices. These vortices are currently being investigated.

The experiment was performed in the U.C.L.A. Geophysical Fluid Dynamics Laboratory, and the author would like to thank Professor W. V. R. Malkus and Mr Paul Cox for assistance.

This work was supported by the National Science Foundation under grant no. GA-849.

REFERENCES

Busse, F. 1968 *J. Fluid Mech.* **33**, 739.

Erdelyi, A., Magnus, W., Oberhettinger, F. & Tricomi, F. G. 1953 *Higher transcendental functions.* New York: McGraw-Hill.

Malkus, W. V. R. 1968 *Science*, **160**, 259.

McEwan, A. D. 1970 *J. Fluid Mech.* **40**, 603.

Wood, W. W. 1965 *J. Fluid Mech.* **22**, 337.

Wood, W. W. 1966 *Proc. Roy. Soc.* A **293**, 181.